

A Direct Evaluation of the Periods of the Weierstrass Zeta Function

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Introduction

The Weierstrass Zeta function

$$Z_L(z) = \frac{1}{z} + \sum_{0 \neq \lambda \in L} \left(\frac{1}{z - \lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \right)$$

of a rank 2 lattice $L \subseteq \mathbb{C}$ is very important for the theory of elliptic functions. Though not elliptic itself, (minus) its derivative

$$\wp_L(z) = \frac{1}{z^2} + \sum_{0 \neq \lambda \in L} \left(\frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right)$$

is elliptic, and together with \wp'_L generates the field of functions which are elliptic with respect to L . The Zeta function Z_L also gives the relation between the Weierstrass σ function, a theta function of the lattice L , and the elliptic function \wp_L , since Z_L is the logarithmic derivative of the σ function and $Z'_L = -\wp_L$. Moreover, like \wp_L is used in order to construct Eisenstein series of weight 2 (and its from derivatives one obtains Eisenstein series of higher weights), the function Z_L is used in order to construct Eisenstein series of weight 1 (see Chapter 4 of [DS]). In fact, the present paper was obtained during the author's study of this construction of weight 1 Eisenstein series appearing in [DS].

Since \wp_L is elliptic, the Zeta function Z_L has a difference function

$$\eta_L : L \rightarrow \mathbb{C}, \quad \eta_L(\lambda) = Z_L(z + \lambda) - Z_L(z),$$

where the latter expression is a constant independent of z . The map η_L is a homomorphism, and can be considered as the period map of \wp with respect to $L = H_1(\mathbb{C}/L, \mathbb{Z})$. These constants $\eta_L(\lambda)$ have various applications in the

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theory of elliptic functions: In particular, one of them appears in the multiplier of the Fourier expansion of the sigma function. In all the references known to the author (see, for example, Chapter 18 of [L]), one has to use various indirect tools (contour integration, etc.) in order to evaluate η_L . Here we show that by choosing a specific order of summation for the series defining the zeta function we can obtain the values of η_L immediately, without the use of any other machinery. The same applies for obtaining the ellipticity of \wp_L and its derivatives, though here the usual proof is also very short and simple.

The value of the Eisenstein series G_2 at $\tau \in \mathcal{H}$, defined by

$$G_2(\tau) = \sum_{c \in \mathbb{Z}} \sum_{d, (c,d) \neq (0,0)} \frac{1}{(c\tau + d)^2}$$

(inner summation on d , then on c), appears in the values of η_L for $L = L_\tau$. One knows that G_2 is quasi-modular of weight 2, but in order to prove this one needs to use various tools, in order to bypass the problem that the defining series for G_2 converges only conditionally. We show how the full quasi-modular behavior of G_2 follows immediately from its relation with η_L and from the homogeneity property of the latter, without even needing to verify that the quasi-modular action is an action.

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1 A Simple Illustration: \wp and its Derivatives

As one sees in every reference, evaluations are easier when one of generators of the lattice $L \subseteq \mathbb{C}$ is 1. For an element $\tau \in \mathcal{H} = \{\tau \in \mathbb{C} | \Im \tau > 0\}$, we have the lattice $L_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$, and the index L_τ is classically replaced simply by τ . \mathbb{C} denotes, of course, the field of complex numbers, and $\Im w$ is the imaginary part of the complex number w .

We start by presenting our argument for the simplest case, of the derivatives of \wp . For $k \geq 3$ we have that the $(k-2)$ th derivative of \wp_L is

$$\wp_L^{(k-2)}(z) = (-1)^k (k-1)! \sum_{\lambda \in L} \frac{1}{(z - \lambda)^k},$$

hence for $L = L_\tau$ we get

$$\wp_\tau^{(k-2)}(z) = (-1)^k (k-1)! \sum_{(c,d) \in \mathbb{Z}^2} \frac{1}{(z - (c\tau + d))^k}.$$

The only identities which we shall need are the following classical equality

$$\frac{1}{w} + \sum_{d=1}^{\infty} \left(\frac{1}{w+d} + \frac{1}{w-d} \right) = \pi \cot \pi w = -\pi i - 2\pi i \sum_{m=1}^{\infty} \mathbf{e}(mw) \quad (1)$$

and its derivatives

$$\sum_{d=-\infty}^{\infty} \frac{1}{(w+d)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} \mathbf{e}(mw), \quad (2)$$

both valid for $w \in \mathcal{H}$ with $\mathbf{e}(\sigma) = e^{2\pi i \sigma}$ for all $\sigma \in \mathbb{C}$ (see, for example, Equations (1.1) and (1.2) of Chapter 1 of [DS]—note that the summation in Equation (1.1) there begins with $m = 0$ while we start with $m = 1$). The left equality in Equation (1) follows by taking the logarithmic derivative (at $z = w$) of the product expansion for the sine function,

$$\sin \pi z = \pi z \prod_{d=1}^{\infty} \left(1 - \frac{z^2}{d^2}\right)$$

(which can be deduced, for example, from the relation between $\sin \pi z$ and the product of two Gamma functions, using the Weierstrass product expansion of the latter). The right equality in Equation (1) is just a geometric expansion of $\pi \cot \pi w = \pi i \left(-1 - \frac{2\mathbf{e}(w)}{1-\mathbf{e}(w)}\right)$. For $w \in \overline{\mathcal{H}}$ we write $w = -(-w)$, or alternatively use the geometric expansion of $\pi \cot \pi w$ as $\pi i \left(1 + \frac{2\mathbf{e}(-w)}{1-\mathbf{e}(-w)}\right)$. Then Equation (1) and gets the form

$$\frac{1}{w} + \sum_{d=1}^{\infty} \left(\frac{1}{w+d} + \frac{1}{w-d} \right) = \pi \cot \pi w = +\pi i + 2\pi i \sum_{m=1}^{\infty} \mathbf{e}(-mw) \quad (3)$$

and the corresponding Equation (2) is obtained by differentiation:

$$\sum_{d=-\infty}^{\infty} \frac{1}{(w+d)^k} = \frac{(+2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} \mathbf{e}(-mw). \quad (4)$$

We want to substitute in these formulae the value $w = z - c\tau$, where τ is the index of the lattice L_τ and z is the argument of the function we investigate. We avoid the poles of the functions by considering $z \notin L_\tau$, but still this value of w can be real (though not integral). For this (and for future use) we recall that for a real number x we have its lower integral value $\lfloor x \rfloor = \max\{n \in \mathbb{Z} | n \leq x\}$, its upper integral value $\lceil x \rceil = \min\{n \in \mathbb{Z} | n \geq x\}$, and its fractional part $\{x\} = x - \lfloor x \rfloor$, the latter being the unique number $0 \leq a < 1$ which lies in $x + \mathbb{Z}$. Then for real w we use Equation (1) or (3) just in the form

$$\frac{1}{w} + \sum_{d=1}^{\infty} \left(\frac{1}{w+d} + \frac{1}{w-d} \right) = \pi \cot \pi w = \pi \cot \pi a, \quad a = \{w\} \quad (5)$$

(with no Fourier expansion). In Equation (2) or (4) we cannot use the Fourier expansion as well, but by decomposing the left hand side of these equations to $d > -w$ and $d < -w$ (again, we do not have $d = -w$ since we assume $w \notin \mathbb{Z}$) one easily sees that we have

$$\sum_{d=-\infty}^{\infty} \frac{1}{(w+d)^k} = \zeta(k, a) + (-1)^k \zeta(k, 1-a), \quad a = \{w\}. \quad (6)$$

Here $\zeta(s, a)$ denotes the Hurwitz zeta function, defined for $\Re s > 1$ and $a > 0$ by the series $\sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$.

Since the sum defining $\wp_{\tau}^{(k-2)}(z)$ converges absolutely, we decide to do the summation first on d and then on c . We use Equation (2) for $c < \frac{\Im z}{\Im \tau}$, (4) for $c > \frac{\Im z}{\Im \tau}$, and in case $\frac{\Im z}{\Im \tau}$ is an integer we use Equation (6) for $c = \frac{\Im z}{\Im \tau}$ (we assume $z \notin L_{\tau}$), where $w = z - c\tau$. We use the classical notation $q_{\tau} = \mathbf{e}(\tau)$ and $q_z = \mathbf{e}(z)$, hence we have $\mathbf{e}(mw) = q_z^m q_{\tau}^{-cm}$ and $\mathbf{e}(-mw) = q_{\tau}^{cm} q_z^{-m}$ for any $m \in \mathbb{N}$. These substitutions yield that

$$\frac{(-1)^k}{(k-1)!} \wp_{\tau}^{(k-2)}(z) = \sum_c \sum_d \frac{1}{((z - c\tau) - d)^k}$$

equals

$$\frac{(-2\pi i)^k}{(k-1)!} \sum_{c < \frac{\Im z}{\Im \tau}} \sum_{m=1}^{\infty} m^{k-1} q_z^m q_{\tau}^{-cm} + \frac{(+2\pi i)^k}{(k-1)!} \sum_{c > \frac{\Im z}{\Im \tau}} \sum_{m=1}^{\infty} m^{k-1} q_{\tau}^{cm} q_z^{-m}$$

(recall the value of w) plus an element appearing only if $\frac{\Im z}{\Im \tau} \in \mathbb{Z}$. Replacing m by $-m$ in case $c > \frac{\Im z}{\Im \tau}$ and noticing that the sign coming from m^{k-1} in this case and the sign difference between the coefficients $(-2\pi i)^k$ and $(+2\pi i)^k$ combine just to -1 allows us to prove

Proposition 1. *The function $\frac{(-1)^k}{(k-1)!} \wp_{\tau}^{(k-2)}(z)$ equals*

$$\delta \cdot (\zeta(k, a) + (-1)^k \zeta(k, 1-a)) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{c \neq \frac{\Im z}{\Im \tau}} \sum_{m \rho > 0} \text{sgn}(m) m^{k-1} q_z^m q_{\tau}^{-cm},$$

where $\rho = \Im(z - c\tau)$ and δ is 1 if $\Im z$ is an integral multiple c of $\Im \tau$ (and then $a = \{z - c\tau\}$ with $z - c\tau \in \mathbb{R} \setminus \mathbb{Z}$), and is 0 otherwise.

The idea is that the expression in Proposition 1 shows that $\wp_{\tau}^{(k-2)}$ is a lattice function. Indeed, we claim that both parts of this expression are invariant under both translations $z \mapsto z + 1$ and $z \mapsto z + \tau$. For the second part, its invariance under $z \mapsto z + 1$ is immediate since z appears there only via q_z . As for $z \mapsto z + \tau$, it takes $\rho = \Im(z - c\tau)$ to $\Im(z - (c-1)\tau)$, $q_z^m q_{\tau}^{-cm}$ to $q_z^m q_{\tau}^{-(c-1)m}$, and the condition $c \neq \frac{\Im z}{\Im \tau}$ to $c \neq \frac{\Im z}{\Im \tau} + 1$, so that replacing c by $c + 1$ gives the asserted invariance. As for the second part, it is clear that δ is preserved by $z \mapsto z + 1$ and $z \mapsto z + \tau$: Indeed, if $c = \frac{\Im z}{\Im \tau}$ is integral then so are $c = \frac{\Im(z+1)}{\Im \tau}$ and $c + 1 = \frac{\Im(z+\tau)}{\Im \tau}$, and otherwise they are both non-integral. Moreover, in case $\delta = 1$ the further dependence on z is only through $a = \{z - c\tau\}$. Since $z \mapsto z + 1$ adds 1 to the argument of the fractional value and for $z \mapsto z + \tau$ the (real) number $z + \tau - (c+1)\tau$ is the same as $z - c\tau$, this part has the asserted invariance as well. The homogeneity property of $\wp_{\alpha L}^{(k-2)}$, namely

$$\wp_{\alpha L}^{(k-2)}(\alpha z) = \alpha^{-k} \wp_L^{(k-2)}(z)$$

for every lattice $L \subseteq \mathbb{C}$, $z \in \mathbb{C}$, and $0 \neq \alpha \in \mathbb{C}$, now shows that $\wp_L^{(k-2)}$ is an elliptic function for any lattice L . Proposition 1 with $k = 3$ is related to the Fourier expansion of $\wp'_\tau(z)$ given in Proposition 3 of Section 2 in Chapter 4 of [L], from which one can also obtain the ellipticity of \wp'_τ (hence of \wp'_L for any L by homogeneity).

For $\wp_L^{(k-2)}$ we did not need all this, since its ellipticity is clear from its defining series. However, for \wp itself this is not so obvious. Indeed, the ellipticity of \wp follows immediately from that of \wp' and the fact that \wp is an even function of z , but it is nice to see (before we go to the more complicated case of Z) how it can also be obtained using our argument. When we specialize the definition of $\wp_L(z)$ to $L = L_\tau$ we obtain

$$\wp_\tau(z) = \frac{1}{z^2} + \sum_{(c,d) \neq (0,0)} \left(\frac{1}{(z - (c\tau + d))^2} - \frac{1}{(c\tau + d)^2} \right),$$

and we again do the summation first over d and then over c . Then both parts of the sum converge, and we get

$$\wp_\tau(z) = \sum_{c \in \mathbb{Z}} \sum_{d \in \mathbb{Z}} \frac{1}{(z - (c\tau + d))^2} - \sum_{c \in \mathbb{Z}} \sum_{d, (c,d) \neq (0,0)} \frac{1}{(c\tau + d)^2}.$$

The second sum is (by definition) the classical Eisenstein series $G_2(\tau)$ (and is independent of z), and the first one can be evaluated in the same way which led us to Proposition 1. We thus obtain

Proposition 2. *The function $\wp_\tau(z)$ equals*

$$\delta \cdot (\zeta(2, a) + \zeta(2, 1 - a)) + (-2\pi i)^2 \sum_{c \neq \frac{3z}{3\tau}} \sum_{m\rho > 0} \operatorname{sgn}(m) m q_z^m q_\tau^{-cm} - G_2(\tau),$$

with ρ , δ , and a having the same meaning as in Proposition 1.

The same argument we had for the derivatives of \wp_τ yields the ellipticity of \wp_τ , since the additional summand $G_2(\tau)$ is a constant independent of z . The homogeneity property

$$\wp_{\alpha L}(\alpha z) = \alpha^{-2} \wp_L(z)$$

for L , z , and α as above shows the ellipticity of \wp_L for any L , without using the parity of \wp_L . Also here, two other forms of the Fourier expansion of $\wp_\tau(z)$ appearing in Proposition 2 can be found in Propositions 2 and 3 of Section 2 in Chapter 4 of [L], and from the latter one can deduce the ellipticity of \wp_τ (thus, by homogeneity, also of \wp_L for any L) as with our argument.

The expression $G_2(\tau)$ is seen to be convergent by precisely the same method, yielding (using some symmetry conditions) the well-known Fourier expansion

$$G_2(\tau) = 2\zeta(2) + 2(2\pi i)^2 \sum_{c=1}^{\infty} \sum_{m=1}^{\infty} m q_\tau^{cm} = \frac{\pi^2}{3} - 8\pi^2 \sum_n \sigma_1(n) q_\tau^n$$

with $\sigma_1(n) = \sum_{d|n} d$ (see, for example, Chapter 1 of [DS]). The convergence properties of this expansion shows that G_2 is holomorphic (and also invariant under $\tau \mapsto \tau + 1$), but nothing more at this point.

2 The Weierstrass Zeta Function

Until now we have only replaced easy proofs of certain assertions by longer proofs of these assertions. We now show how by using the same ideas we can evaluate η_τ easily (with no integration), and the Legendre relation just follows from evaluation (rather than being a tool for the proof).

We specialize $Z_L(z)$ to $L = L_\tau$ in order to obtain

$$Z_\tau(z) = \frac{1}{z} + \sum_{(c,d) \neq (0,0)} \left(\frac{1}{z - (c\tau + d)} + \frac{1}{c\tau + d} + \frac{z}{(c\tau + d)^2} \right)$$

(which is known to be an absolutely convergent sum), and we fix the following summation order: Again we sum first over d and then over c , and in each of them we take first the index 0, and then the summands of d and $-d$ (in the inner sum) or c and $-c$ (in the outer sum) together. We then write $Z_\tau(z)$ as the sum of three infinite series, each corresponding to an element in the parentheses in the latter expression, namely

$$\begin{aligned} \frac{1}{z} + \sum_{f=1}^{\infty} \left(\frac{1}{z+f} + \frac{1}{z-f} \right) + \sum_{e=1}^{\infty} \left(\frac{1}{z+e\tau} + \sum_{f=1}^{\infty} \left(\frac{1}{z+e\tau+f} + \frac{1}{z+e\tau-f} \right) + \right. \\ \left. + \frac{1}{z-e\tau} + \sum_{f=1}^{\infty} \left(\frac{1}{z-e\tau+f} + \frac{1}{z-e\tau-f} \right) \right) \end{aligned} \quad (7)$$

from the first summand, and similar series from the second and third summands. We claim that all three series converge (in this order of summation). Indeed, the second series begins with $\sum_{f=1}^{\infty} \left(\frac{1}{f} + \frac{1}{-f} \right) = 0$ and continues with

$$\sum_{e=1}^{\infty} \left(\frac{1}{e\tau} + \sum_{f=1}^{\infty} \left(\frac{1}{e\tau+f} + \frac{1}{e\tau-f} \right) + \frac{1}{-e\tau} + \sum_{f=1}^{\infty} \left(\frac{1}{-e\tau+f} + \frac{1}{-e\tau-f} \right) \right)$$

which also vanishes. The order of summation of the third series yields directly $zG_2(\tau)$. The convergence of the first series, i.e., the one appearing in Equation (7), follows from the fact that the sum of all three converges to $Z_\tau(z)$, though we now want to evaluate it explicitly as well.

Let us now use our usual argument in order to evaluate the sum in Equation (7). Note that the first part of it is just the left hand side of Equation (1), (3) or (5) for $w = z$, and for each e what we have is the sum of the left hand side of Equation (1), (3) or (5) with $w = z+e\tau$ and with $w = z-e\tau$. We thus substitute the right hand side of the corresponding equation (and value of w), which give

us expressions similar to what we encountered in the argument which proved Proposition (1) and (2). However, there are two differences. First, for integral $\frac{\Im z}{\Im \tau}$ we now have $\pi \cot \pi a$ rather than the Hurwitz zeta function. Second, and more important, is the constant, $-\pi i$ in Equation (1) and $+\pi i$ in Equation (3), which we have to add to the power series in $\mathbf{e}(\pm w)$. The part involving q_z and q_τ converges as with \wp_τ and its derivatives, and we have to see that the constants also converge in the chosen order of summation. What assures us that they do is the fact that for $e > \frac{|\Im z|}{\Im \tau}$, $z + e\tau$ lies in \mathcal{H} and $z - e\tau$ lies in $\overline{\mathcal{H}}$, so that we use Equation (1) for the first and Equation (3) for the second, and the constants cancel. Hence for all but finitely many e the constants cancel, and the rest give some integral multiple of $-\pi i$. This proves

Proposition 3. *We have*

$$Z_\tau(z) = -t\pi i + \delta \cdot \pi \cot \pi a - 2\pi i \sum_{c \neq \frac{\Im z}{\Im \tau}} \sum_{m \rho > 0} \operatorname{sgn}(m) q_z^m q_\tau^{-cm} + z G_2(\tau),$$

with ρ , δ , and a having the same meaning as in Propositions 1 and 2, and t is some (finite) integer depending on τ and z .

Indeed, as we have seen, a finite number of e -summands (possibly together with the first one, corresponding to $e = 0$) involve a constant $\mp \pi i$ from Equations (1) and (3), and we remember to include the contribution of the third summands in the series defining $Z_\tau(z)$. We remark that Proposition 2 can be obtained directly from Proposition 3 by differentiation with respect to z (for doing this carefully, we differentiate with respect to the real part, noting that δ depends only on $\Im z$ and, as Proposition 4 below shows, the same holds for t). Similarly, one can deduce Proposition 1 by a $(k - 2)$ -fold differentiation of the equality appearing in Proposition 2 with respect to z . In relation with our Proposition 3, we note the existence of Equation (1) in Section 3 of Chapter 18 of [L] (and another formula appearing right after it), also giving a Fourier expansion of $Z_\tau(z)$ up to a factor which is linear in z . Then $\eta_\tau(1)$ is evaluated by some Fourier series in q_τ in Equation (2) of that Section of [L] (which equals $G_2(\tau)$ by Exercise 4.8.3 of [DS], for example).

We are going to deduce the values of η_τ from the expansion of $Z_\tau(z)$ as in Proposition 3 (and the evaluation of t in Proposition 4). In this context we remark that doing this from the equations in [L] as in our argument is much harder, since the linear function there has the coefficient $\eta_\tau(1)$. Comparing the equations of [L] (which one obtains after some work there) with our Proposition 3 yields immediately the value of $\eta_\tau(1)$ as $G_2(\tau)$, and if one proves the Legendre relation indepenendtly (as in Section 1 of Chapter 18 of [L] or Exercise 4.8.2 of [DS]), the value of $\eta_\tau(\tau)$ (hence the general formula for η_τ) follows. However, we stick to our simple, almost prerequisite-free, approach, and continue using Proposition 3 alone.

It remains to find out the value of the integer t , where we recall that only $e \leq \frac{|\Im z|}{\Im \tau}$ have to be considered. For $z \notin \mathbb{R}$ the first sum (without e) contributes

$\text{sgn}(\Im z)$ to t , while for $z \in \mathbb{R}$ there is no contribution at all. Indeed, for $\Im z > 0$ we use Equation (1) with the constant $-\pi i$, for $\Im z < 0$ we take Equation (3) with $+\pi i$ (recall that t is the coefficient of $-\pi i$), and in Equation (5), used for $z \in \mathbb{R}$, there is no contribution to t . For every $e < \frac{|\Im z|}{\Im \tau}$ (this can be an empty set of integers, as is the case where $-\Im \tau \leq \Im z \leq \Im \tau$) both $z + e\tau$ and $z - e\tau$ have the same sign of imaginary part as z , so that each such e contributes $2\text{sgn}(\Im z)$ to t (by the same argument). In the case where $\frac{\Im z}{\Im \tau}$ is a non-zero integer we see that for the value $e = \frac{|\Im z|}{\Im \tau}$ one summand gives real w (and no contribution to the constant) and the other gives a contribution of $\text{sgn}(\Im z)$ to t as above. This is the basis of the proof of

Proposition 4. *The integer t from Proposition 3 is given by*

$$t = \left\lfloor \frac{\Im z}{\Im \tau} \right\rfloor + \left\lceil \frac{\Im z}{\Im \tau} \right\rceil = \left\lfloor \frac{\Im z}{\Im \tau} \right\rfloor - \left\lfloor \frac{-\Im z}{\Im \tau} \right\rfloor.$$

Proof. Write $x = \frac{\Im z}{\Im \tau}$, and we begin by assuming the x is not an integer. Then the number of e -summands which have a contribution of 2 is $\lfloor |x| \rfloor$, so that the above argument yields $t = (2\lfloor |x| \rfloor + 1)\text{sgn}(x)$. For positive x this is just $2\lfloor x \rfloor + 1 = \lfloor x \rfloor + \lceil x \rceil$ (x is not an integer, so that $\lceil x \rceil = \lfloor x \rfloor + 1$), while for negative x we write $\lfloor -x \rfloor = -\lceil x \rceil$ and this is $-2\lfloor -x \rfloor - 1 = 2\lceil x \rceil - 1 = \lfloor x \rfloor + \lceil x \rceil$ (we still assume $x \notin \mathbb{Z}$). This shows that t has the asserted middle expression in this case. Next consider the case $0 \neq x \in \mathbb{Z}$ is a nonzero integer. In this case the number of e -summands which have a contribution of 2 is $|x| - 1$ (since we had a sharp inequality on e there), but since we had an extra contribution of 2 (one from z , one from $e = |x|$) we get $t = 2|x|\text{sgn}(x) = 2x$. Since for integral x we have $\lfloor x \rfloor = \lceil x \rceil = x$ this also agrees with the asserted middle expression. For $x = 0$ (i.e., real z) we have no constant contribution at all, and the value $t = 0$ is indeed the asserted middle expression for $x = 0$. This covers all the possible cases, hence proves the validity of the middle expression. The expression on the right is seen to equal the middle expression there by applying the identity $\lfloor -x \rfloor = -\lceil x \rceil$ again, which completes the proof of the proposition. \square

We will later use only the middle expression in the equality appearing in Proposition 4, but the expression on the right hand side is included since it has the advantage of using only the (more intuitive) lower integral value function. Moreover, it reflects better the fact that Z_τ is an odd function of z .

We can now evaluate the lattice function η_τ directly from Propositions 3 and 4, as given in the following

Theorem 5. *The difference function η_τ is given on the generators 1 and τ of L_τ by*

$$\eta_\tau(1) = G_2(\tau), \quad \eta_\tau(\tau) = \tau G_2(\tau) - 2\pi i.$$

For a general element $\lambda = c\tau + d$ of L_τ we have

$$\eta_\tau(c\tau + d) = (c\tau + d)G_2(\tau) - 2\pi ic.$$

Proof. Since t depends only in $\Im z$ and the other sum depends on z only through q_z , it follows that the translation $z \mapsto z + 1$ adds to $Z_\tau(z)$ only the $G_2(\tau)$ from the latter summand. This proves the value of $\eta_\tau(1)$. As for the $z \mapsto z + \tau$, in the middle sum we have the usual summation index change (which has no effect on the value), while t is increased by 2. Indeed, $\frac{\Im z}{\Im \tau}$ increases by 1, hence so do the lower and upper integral values. Taking also the last summand into consideration, we obtain the asserted value of $\eta_\tau(\tau)$. The expression for the general value $\eta_\tau(c\tau + d)$ can be obtained the additivity of η_τ . Alternatively, we can deduce the general value directly from Propositions 3 and 4: The part of Proposition 3 with $G_2(\tau)$ gives the first summand, the series appearing in Proposition 3 is invariant under any change $z \mapsto z + \lambda$ for $\lambda \in L_\tau$, and the value of t is increased by $2c$. Then the values of $\eta_\tau(1)$ and $\eta_\tau(\tau)$ are just special cases of the general formula. \square

We recall the homogeneity property of Z , from which a similar one holds for η , namely

$$Z_{\alpha L}(\alpha z) = \alpha^{-1} Z_L(z), \quad \eta_{\alpha L}(\alpha \lambda) = \alpha^{-1} \eta_L(\lambda)$$

for L , z , and α as above and $\lambda \in L$. This implies that for a general lattice $L = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$ normalized such that $\tau = \frac{w_1}{w_2}$ is in \mathcal{H} , we have

$$\eta_L(w_2) = \frac{G_2(\tau)}{w_2}, \quad \eta_L(w_1) = \frac{\tau G_2(\tau) - 2\pi i}{w_2}.$$

This is so since $L = w_2 L_\tau$. We therefore obtain

Corollary 6. *The Legendre relation holds:*

$$w_1 \eta_L(w_2) - w_2 \eta_L(w_1) = +2\pi i.$$

Indeed, one just substitutes the value of τ and obtain the equality in Corollary 6. Alternatively, the special case of Corollary 6 with $L = L_\tau$ with the basis $w_2 = 1$ and $w_1 = \tau$ follows immediately from Theorem 5, and then the homogeneity property of η_L , which is compensated by the (trivial) homogeneity of the coefficients, extends the validity of Corollary 6 to any lattice L and a normalized basis. We remark again that in [L] (as well as in other references dealing with evaluating the function η_τ), one first uses integration in order to obtain the Legendre relation in Corollary 6, then one evaluates (by more difficult means) $\eta_\tau(1)$, and only then the value of $\eta_\tau(\tau)$ (hence of $\eta_\tau(\lambda)$ for any $\lambda \in L_\tau$) follows. In our approach, we get all the values of $\eta_\tau(\lambda)$ at once, and Corollary 6 is proved by a simple substitution of values.

3 Quasi-Modularity of G_2

At this point, all that we know about the Eisenstein series G_2 is that it is holomorphic and invariant under $\tau \mapsto \tau + 1$. We now use its relation with η_τ and the homogeneity of the latter in order to obtain its quasi-modular behavior under the action of $SL_2(\mathbb{Z})$.

Theorem 7. *If the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $SL_2(\mathbb{Z})$ then we have*

$$G_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 G_2(\tau) - 2\pi ic(c\tau + d).$$

Proof. If we know that the mapping $(M, \tau) \mapsto \frac{a\tau + b}{c\tau + d}$ for M as above defines an action of $SL_2(\mathbb{Z})$ on \mathcal{H} , that $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ generate $SL_2(\mathbb{Z})$, and that if the asserted relation holds for two elements M and N then it holds for their product, then we can argue as follows. We already have the invariance under $T : \tau \mapsto \tau + 1$. Writing L_τ as $\tau \cdot L_{-1/\tau}$ gives us, together with Theorem 5 and the homogeneity property of η_L ,

$$\tau G_2(\tau) - 2\pi i = \eta_\tau(\tau) = \frac{\eta_{-1/\tau}(1)}{\tau} = \frac{G_2(-\frac{1}{\tau})}{\tau},$$

which gives us the required behavior of G_2 under the action of S on \mathcal{H} , which is $\tau \mapsto -\frac{1}{\tau}$. This bypasses the need to deal with the conditional convergence, and gives the desired relation for S immediately. Then one uses the facts mentioned in the beginning of this paragraph in order to obtain the assertion for any element M of $SL_2(\mathbb{Z})$.

However, we can obtain the asserted relation without needing to know any of these facts. Observe that the fact that $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $SL_2(\mathbb{Z})$ implies that L_τ is generated by $w_1 = a\tau + b$ and $w_2 = c\tau + d$, and they are normalized in this order. Write $M\tau = \frac{a\tau + b}{c\tau + d} \in \mathcal{H}$, and this shows that $L_\tau = (c\tau + d)L_{M\tau}$, whence (as we did for $M = S$ above) the general formula in Theorem 5 and the homogeneity property of η_L yield the equality

$$(c\tau + d)G_2(\tau) - 2\pi ic = \eta_\tau(c\tau + d) = \frac{\eta_{M\tau}(1)}{c\tau + d} = \frac{G_2(M\tau)}{c\tau + d}.$$

This proves the asserted relation between $G_2(M\tau)$ and $G_2(\tau)$ without needing to base on any other results. \square

References

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